Comonadic Account of Feferman-Vaught-Mostowski Theorems

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October 23, 2022

The setting

Let σ be a set of relational symbols with positive arities, we can define a category of σ-structures $\mathcal{R}(\sigma)$:

- Objects are $\mathcal{A} = (A, \{R^{\mathcal{A}}\}_{R \in \sigma})$ where $R^{\mathcal{A}} \subseteq A^r$ for r-ary relation symbol R.
- Morphisms $f : \mathcal{A} \to \mathcal{B}$ are relation preserving set functions $f : A \to B$

$$R^{\mathcal{A}}(a_1, \ldots, a_r) \Rightarrow R^{\mathcal{B}}(f(a_1), \ldots, f(a_r))$$

- Embeddings $f \colon A \rightarrowtail B$ are injective functions which reflect relations:

$$R^{\mathcal{A}}(a_1, \ldots, a_r) \Leftarrow R^{\mathcal{B}}(f(a_1), \ldots, f(a_r))$$

Setting for graph theory, database theory, and descriptive complexity

Category theory vs. model theory

Category theorists look at structures "as they really are"; i.e. up to isomorphism $\mathcal{A} \cong \mathcal{B}$

Model theorists look at structures with the "blurry lens" of a logic $\mathcal{L}$:

$$\mathcal{A} \equiv^{\mathcal{L}} \mathcal{B} := \forall \phi \in \mathcal{L}, \mathcal{A} \vDash \phi \Leftrightarrow \mathcal{B} \vDash \phi$$

$$\mathcal{A} \cong \mathcal{B} \Rightarrow \mathcal{A} \equiv^{\mathcal{L}} \mathcal{B}$$

$$\mathcal{A} \Rrightarrow^{\mathcal{L}} \mathcal{B} := \forall \phi \in \mathcal{L}, \mathcal{A} \vDash \phi \Rightarrow \mathcal{B} \vDash \phi$$

For a logic $\mathcal{J}$, $\equiv_{\mathcal{J}}$ may satisfy Feferman-Vaught-Mostowski (FVM) theorems

If $\mathcal{A}_i \equiv_{\mathcal{J}} \mathcal{B}_i$ for all $i \in I$, then

- Products: $\mathcal{A}_1 \times \mathcal{A}_2 \equiv_{\mathcal{J}} \mathcal{B}_1 \times \mathcal{B}_2$ and $\prod_i \mathcal{A}_i \equiv_{\mathcal{J}} \prod_i \mathcal{B}_i$
- Coproducts: $\mathcal{A}_1 + \mathcal{A}_2 \equiv_{\mathcal{J}} \mathcal{B}_1 + \mathcal{B}_2$ and $\coprod_i \mathcal{A}_i \equiv_{\mathcal{J}} \coprod_i \mathcal{B}_i$

For an operation $H \colon \mathcal{C}_1 \times \mathcal{C}_2 \cdots \times \mathcal{C}_n \to \mathcal{D}$ and logics $\mathcal{J}_1, \ldots, \mathcal{J}_n, \mathcal{J}$:

$$\mathcal{A}_i \equiv^{\mathcal{J}_i} \mathcal{B}_i \text{ implies } H(\mathcal{A}_1, \ldots, \mathcal{A}_n) \equiv^{\mathcal{J}} H(\mathcal{B}_1, \ldots, B_n)$$

with $\mathcal{A}_i, \mathcal{B}_i \in \mathcal{C}_i$ where $\mathcal{C}_i, \mathcal{D}$ are relevant categories of models.

Key ingredient in Courcelle's theorems and other algorithmic metatheorems

How can we prove such statements categorically?

- In every round i, of the k-round game $\mathbf{EF}_k(\mathcal{A}, \mathcal{B})$:
  - Spoiler chooses an element $a_i \in \mathcal{A}$ or $b_i \in \mathcal{B}$
  - Duplicator responds with $b_i \in \mathcal{B}$ or $a_i \in \mathcal{A}$

  Duplicator wins the round if the relation $\gamma_i = \{(a_j, b_j) \mid j \leq i\}$ is a partial isomorphism

Theorem

Duplicator has a winning strategy in $\mathbf{EF}_k(\mathcal{A}, \mathcal{B})$ *iff* $\mathcal{A} \equiv_{\mathcal{L}_k} \mathcal{B}$

One-sided variant: $\mathcal{A} \Rrightarrow_{\exists^+\mathcal{L}_k} \mathcal{B}$. No alternation between structures. Partial homomorphism

Bijection variant: $\mathcal{A} \equiv_{\#\mathcal{L}_k} \mathcal{B}$. Duplicator chooses a bijection before Spoiler's choice and responds using bijection

$\#\mathcal{L}_k$ has quantifiers of the from $\exists_{\leq n} x, \exists_{\geq n} x$

Given a σ-structure $\mathcal{A}$, we can create σ-structure $\mathbb{E}_k\mathcal{A}$ on non-empty sequences of elements in A of length $\leq k$

Let $\varepsilon_{\mathcal{A}} : \mathbb{E}_k\mathcal{A} \to \mathcal{A}$ return the last move of the play $[a_1, \ldots, a_n] \mapsto a_n$.

$$R^{\mathbb{E}_k\mathcal{A}}(s_1, \ldots, s_r) \Leftrightarrow s_i \sqsubseteq s_j \text{ or } s_j \sqsubseteq s_i \text{ for } i, j \in [\mathsf{r}] \\ \text{and } R^{\mathcal{A}}(\varepsilon_{\mathcal{A}}(s_1), \ldots, \varepsilon_{\mathcal{A}}(s_r))$$

Comultiplication $\delta\colon \mathbb{E}_k\mathcal{A} \to \mathbb{E}_k\mathbb{E}_k\mathcal{A}$ where

$$\delta([a_1, \ldots, a_n]) = [[a_1], [a_1, a_2], \ldots, [a_1, \ldots, a_n]]$$

Kleisli category $\mathbf{Kl}(\mathbb{E}_k)$ for $\mathbb{E}_k$, objects as $\mathcal{R}(\sigma)$, morphisms of type $f\colon \mathbb{E}_k A \to B$, composition $g \cdot f = g \circ \mathbb{E}_k(f) \circ \delta_{\mathcal{A}}$, identity ε_A

Theorem (Abramsky+S 21)

- $\mathcal{A} \to_{\mathbb{E}_k} \mathcal{B} \Leftrightarrow \mathcal{A} \Rrightarrow_{\exists^+\mathcal{L}_k} \mathcal{B}$
- $\mathcal{A} \cong_{\mathbf{Kl}(\mathbb{E}_k)} \mathcal{B} \Leftrightarrow \mathcal{A} \equiv_{\#\mathcal{L}_k} \mathcal{B}$ *(with $\mathcal{A}, \mathcal{B}$ finite)*

$\mathcal{A} \to_{\mathbb{E}_k} \mathcal{B}$ means there exists a coKleisli morphism $f\colon \mathbb{E}_k\mathcal{A} \to \mathcal{B}$

$\mathcal{A} \cong_{\mathbf{Kl}(\mathbb{E}_k)} \mathcal{B}$ means there exists a coKleisli isomorphism between $\mathcal{A}$ and $\mathcal{B}$

$\exists^+\mathcal{L}_k$ and $\#\mathcal{L}_k$ as logics *without equality*.

Universe of $\mathcal{A}_1 \uplus \mathcal{A}_2 = \{(i, a_i) \mid i = \{1,2\}, a_i \in A_i\}$ and relations defined in obvious way

$$R^{\mathcal{A}_1 \uplus \mathcal{A}_2}((i_1, a_1), \ldots, (i_n, a_n)) \Leftrightarrow \exists i \in \{1,2\} \forall j \in [n], i_j = i \\ \text{and } R^{\mathcal{A}_i}(a_1, \ldots, a_n)$$

For every $\mathcal{A}_1, \mathcal{A}_2$ there are

$$\kappa_{\mathcal{A}_1,\mathcal{A}_2} \colon \mathbb{E}_k(\mathcal{A}_1 \uplus \mathcal{A}_2) \to \mathbb{E}_k \mathcal{A}_1 \uplus \mathbb{E}_k \mathcal{A}_2$$

$$\kappa([(i_1, a_1), \ldots, (i_n, a_1)] = \begin{cases} [a_j \mid i_j = 1] & \text{if } i_n = 1 \\ [a_j \mid i_j = 2] & \text{if } i_n = 2 \end{cases}$$

If $\mathcal{A}_i \to_{\mathbb{E}_k} \mathcal{B}_i$, then $f_i \colon \mathbb{E}_k\mathcal{A}_i \to \mathcal{B}_i$ and $g_i \colon \mathbb{E}_k\mathcal{B}_i \to \mathcal{A}_i$ for $i \in \{1, 2\}$

$$\mathbb{E}_k(\mathcal{A}_1 \uplus \mathcal{A}_2) \xrightarrow{\kappa_{\mathcal{A}_1,\mathcal{A}_2}} \mathbb{E}_k\mathcal{A}_1 \uplus \mathbb{E}_k\mathcal{A}_2 \xrightarrow{f_1 \uplus f_2} \mathcal{B}_1 \uplus \mathcal{B}_2$$

So $\mathcal{A}_1 \uplus \mathcal{A}_2 \to_{\mathbb{E}_k} \mathcal{B}_1 \uplus \mathcal{B}_2$ and $\mathcal{A}_1 \uplus \mathcal{A}_2 \Rrightarrow_{\exists^+\mathcal{L}_k} \mathcal{B}_1 \uplus \mathcal{B}_2$

For $\equiv_{\#\mathcal{L}_k}$: if f_i, g_i are inverses for $i \in \{1, 2\}$, then $f_1 \uplus f_2 \circ \kappa, g_1 \uplus g_2 \circ \kappa$ are inverses.

Follows from $\kappa \colon \mathbb{E}_k(\mathcal{A}_1 \uplus \mathcal{A}_2) \to \mathbb{E}_k\mathcal{A}_1 \uplus \mathbb{E}_k\mathcal{A}_2$ being coKleisli law

$$\varepsilon_{\mathcal{A}_1} \uplus \varepsilon_{\mathcal{A}_2} = \kappa \circ \varepsilon_{\mathcal{A}_1 \uplus \mathcal{A}_2} \qquad \delta_{\mathcal{A}_1} \uplus \delta_{\mathcal{A}_2} \circ \kappa = \kappa \circ \mathbb{E}_k\kappa \circ \delta_{\mathcal{A}_1 \uplus \mathcal{A}_2}$$

Theorem

Given

- *operation* $H\colon C_1 \times \cdots \times C_n \to D$,
- *comonads* $\mathbb{C}_1, \ldots, \mathbb{C}_n, \mathbb{D}$ *capturing logics* $\mathcal{J}_1, \ldots, \mathcal{J}_n, \mathcal{J}$
- *coKleisli law* $\kappa\colon \mathbb{D}(H(A_1, \ldots, A_n)) \to H(\mathbb{C}_1(A_1), \ldots, \mathbb{C}_n(A_n))$

Then:

$$\mathcal{A}_i \Rrightarrow_{\exists^+\mathcal{J}_i} \mathcal{B}_i \text{ implies } H(\mathcal{A}_1, \ldots, \mathcal{A}_n) \Rrightarrow_{\exists^+\mathcal{J}} H(\mathcal{B}_1, \ldots, \mathcal{B}_n)$$

$$\mathcal{A}_i \equiv_{\#\mathcal{J}_i} \mathcal{B}_i \text{ implies } H(\mathcal{A}_1, \ldots, \mathcal{A}_n) \equiv_{\#\mathcal{J}} H(\mathcal{B}_1, \ldots, \mathcal{B}_n)$$

Eilenberg-Moore category of coalgebras

Define semantics for $\mathcal{L}_k$ in terms of $\mathbf{EM}(\mathbb{E}_k)$

Coalgebras are morphisms $\alpha : \mathcal{A} \to \mathbb{E}_k\mathcal{A}$ satisfying the equations:

$$\epsilon_{\mathcal{A}} \circ \alpha = \mathsf{id}_{\mathcal{A}} \qquad \mathbb{E}_k\alpha \circ \alpha = \delta_{\mathcal{A}} \circ \alpha$$

with $\delta_{\mathcal{A}} = \mathsf{id}^*_{\mathbb{E}_k\mathcal{A}} : \mathbb{E}_k\mathcal{A} \to \mathbb{E}_k\mathbb{E}_k\mathcal{A}$

We can define the Eilenberg-Moore category $\mathbf{EM}(\mathbb{E}_k)$:

- Objects are coalgebras $(\mathcal{A}, \alpha : \mathcal{A} \to \mathbb{E}_k\mathcal{A})$
- Morphisms are commuting squares:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\alpha} & \mathbb{E}_k\mathcal{A} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \mathbb{E}_k f} \\ \mathcal{B} & \xrightarrow{\beta} & \mathbb{E}_k\mathcal{B} \end{array}$$

Will write $f : \alpha \to \beta$ for a commuting square as above.

$\mathbf{EM}(\mathbb{E}_k)$ represent forest-shaped covers of objects in $\mathcal{R}(\sigma)$ of height $\leq k$

Cofree coalgebra functor $G^{\mathbb{E}_k} : \mathcal{R}(\sigma) \to \mathbf{EM}(\mathbb{E}_k)$ where $\mathcal{A} \mapsto (\mathbb{E}_k\mathcal{A}, \delta_{\mathcal{A}})$

For $(\mathcal{A}, \alpha \colon \mathcal{A} \to \mathbb{E}_k\mathcal{A})$, we obtain an order $\sqsubseteq_\alpha$ on $\mathcal{A}$ compatible with the relations

$$a \sqsubseteq_\alpha a' \Leftrightarrow \alpha(a) \text{ is prefix of } \alpha(a')$$

Subcategory of paths $(P, \pi) \in \mathcal{P} \subseteq \mathbf{EM}(\mathbb{E}_k)$ where $\sqsubseteq_\pi$ is a finite chain

Embeddings $(P, \pi) \rightarrowtail (\mathcal{A}, \alpha)$ pick out paths, and $(P, \pi) \rightarrowtail G^{\mathbb{E}_k}(\mathcal{A})$ pick out plays.

$\mathcal{A} \leftrightarrow_{\mathbb{E}_k} \mathcal{B}$ if there exists a span in $\mathbf{EM}(\mathbb{E}_k)$

$$G^{\mathbb{E}_k}(\mathcal{A}) \xleftarrow{f} (X, \chi) \xrightarrow{g} G^{\mathbb{E}_k}(\mathcal{B})$$

where f, g are

- Pathwise embeddings $e\colon (P, \pi) \rightarrowtail (X, \chi)$ implies $f \circ e\colon (P, \pi) \rightarrowtail G^{\mathbb{E}_k}(\mathcal{A})$
- Open maps, a path which can be extended in the codomain, can be extended in the domain.

Theorem (Abramsky+S 21)

$$\mathcal{A} \leftrightarrow_{\mathbb{E}_k} \mathcal{B} \Leftrightarrow \mathcal{A} \equiv_{\mathcal{L}_k} \mathcal{B}$$

where $\mathcal{L}_k$ is first-order logic up to quantifier rank $\leq k$ without equality.

Compute a span of right type to obtain a FVM theorem for $\uplus$ and $\equiv_{\mathcal{L}_k}$ need a lifting $\hat{\uplus}\colon \mathbf{EM}(\mathbb{E}_k) \times \mathbf{EM}(\mathbb{E}_k) \to \mathbf{EM}(\mathbb{E}_k)$:

$$\begin{array}{ccccc} & & (X, \chi) & & \\ & \swarrow^{f_1 \hat{\uplus} f_2} & & \searrow^{g_1 \hat{\uplus} g_2} & \\ G^{\mathbb{E}_k}(\mathcal{A}_1) \hat{\uplus} G^{\mathbb{E}_k}(\mathcal{A}_2) & & & & G^{\mathbb{E}_k}(\mathcal{B}_1) \hat{\uplus} G^{\mathbb{E}_k}(\mathcal{B}_2) \\ \| & & & & \| \\ G^{\mathbb{E}_k}(\mathcal{A}_1 \uplus \mathcal{A}_2) & & & & G^{\mathbb{E}_k}(\mathcal{B}_1 \uplus \mathcal{B}_2) \end{array}$$

if f_i, g_i, then $f_1 \uplus f_2, g_1 \uplus g_2$ are OPEs follows from:

(S1) If f_1, f_2 are embeddings, then $f_1 \hat{\uplus} f_2$ is embedding

(S2) $e\colon (P, \pi) \rightarrowtail (\mathcal{A}_1, \alpha_1) \hat{\uplus} (\mathcal{A}_2, \alpha_2)$, then there exists a 'minimal decomposition'

$$e = e_1 \hat{\uplus} e_2 \circ e_0$$

where $e_i\colon (P_i, \pi_i) \rightarrowtail (\mathcal{A}_i, \alpha_i)$ for $i \in \{1, 2\}$

Compute $(\mathcal{A}_1, \alpha_1)\hat{\uplus}(\mathcal{A}_2, \alpha_2)$ as the equaliser in $\mathbf{EM}(\mathbb{E}_k)$

$$(\mathcal{A}_1, \alpha_1)\hat{\uplus}(\mathcal{A}_2, \alpha_2) \xrightarrow{\iota} G(\mathcal{A}_1 \uplus \mathcal{A}_2) \underset{F(\alpha_1 \uplus \alpha_2)}{\overset{G(\kappa)\circ\delta}{\rightrightarrows}} G(\mathbb{E}_k\mathcal{A}_1 \uplus \mathbb{E}_k\mathcal{A}_2)$$

- Take the cofree structure $G^{\mathbb{E}_k}(\mathcal{A}_1 \uplus \mathcal{A}_2)$
- Substructure compatible with $(\mathcal{A}_i, \alpha_i)$ and κ, i.e. the words $[(i_1, a_1), \ldots, (i_n, a_n)] \in G(\mathcal{A}_1 \uplus \mathcal{A}_2)$:

$$[a_j \mid i_j = 1] \in \mathbf{im}(\alpha_1) \text{ if } i_n = 1 \quad [a_j \mid i_j = 2] \in \mathbf{im}(\alpha_2) \text{ if } i_n = 2$$

$\hat{\uplus}$ is a 'interleaving' sum of paths in $(\mathcal{A}_1, \alpha_1)$ and $(\mathcal{A}_2, \alpha_2)$

Diagram is (sort-of) dual to the quotient construction of a tensor product of vector spaces $V \otimes W$

Suppose V_1 and V_2 are two finite-dimensional $\mathbb{C}$-vector spaces represented as algebras over $\mathcal{M}_\mathbb{C}$-monad $(V_i, \nu_i \colon \mathcal{M}_\mathbb{C}(V_i) \to V_i)$

$$F(\mathcal{M}_\mathbb{C}(V_1) \times \mathcal{M}_\mathbb{C}(V_2)) \underset{F(\nu_1 \times \nu_2)}{\overset{\mu \circ F(\tau)}{\rightrightarrows}} F(V_1 \times V_2) \xrightarrow{\pi} (V_1, \nu_1) \otimes (V_2, \nu_2)$$

$\tau \colon \mathcal{M}_\mathbb{C}(V_1) \times \mathcal{M}_\mathbb{C}(V_2) \to \mathcal{M}_\mathbb{C}(V_1 \times V_2)$ is a Kleisli law of $\times$ over $\mathcal{M}_\mathbb{C}$ with $\tau(\Sigma_i a_i v_i, \Sigma_j b_j w_j) = \Sigma_{i,j} a_i b_j (v_i, w_j)$

- Take the free vector space $F(V_1 \times V_2)$
- Quotient structure compatible with (V_i, ν_i) and τ

$$(v + w, v') \sim (v, v') + (w, v') \quad (sv, v') \sim s(v, v')$$

$$(v, v' + w') \sim (v, v') + (v, w') \quad (v, sv') \sim s(v, v')$$

Tensor product $\otimes$ is the lifting of $\times$ on **Set** to $\mathbf{Vect}_{\mathbb{C}} \cong \mathbf{EM}(\mathcal{M}_{\mathbb{C}})$ where 'bilinearity' arises from τ.

Interleaving sum $\hat{\uplus}$ is the lifting of $\uplus$ on $\mathcal{R}(\sigma)$ to $\mathbf{EM}(\mathbb{E}_k)$ where the 'interleaving' arises from κ.

"Arising from" means via universal property of a (co)equaliser involving the (co)Kleisli law and the unlifted operation.

We can rephrase this universal property

(V_1, ν_1), (V_2, ν_2), and (W, ξ) objects in $\mathbf{Vect}_{\mathbb{C}} \cong \mathbf{EM}(\mathcal{M}_{\mathbb{C}})$

$\nu_1 \otimes \nu_2 \to \xi$	linear map
$V_1 \times V_2 \to W$	bilinear map

Bilinearity means a morphism f in **Set** satisfying:

$$\begin{array}{ccccc}
\mathcal{M}_{\mathbb{C}}(V_1) \times \mathcal{M}_{\mathbb{C}}(V_2) & \xrightarrow{\tau} & \mathcal{M}_{\mathbb{C}}(V_1 \times V_2) & \xrightarrow{\mathcal{M}_{\mathbb{C}}(f)} & \mathcal{M}_{\mathbb{C}}(W) \\
\downarrow{\scriptstyle \nu_1 \times \nu_2} & & & & \downarrow{\scriptstyle \xi} \\
V_1 \times V_2 & & \xrightarrow{\quad f \quad} & & W
\end{array}$$

$(\mathcal{A}_1, \alpha_1)$, $(\mathcal{A}, \alpha_2)$, and $(\mathcal{B}, \beta)$ objects in $\mathbf{EM}(\mathbb{E}_k)$

$\beta \to \alpha_1 \hat{\uplus} \alpha_2$	coalgebra map
$\mathcal{B} \to \mathcal{A}_1 \uplus \mathcal{A}_2$	interleaving map

Interleaving means a morphism h in $\mathcal{R}(\sigma)$ satisfying:

$$\begin{array}{ccccc}
\mathcal{B} & & \xrightarrow{\quad h \quad} & & \mathcal{A}_1 \uplus \mathcal{A}_2 \\
\downarrow{\scriptstyle \beta} & & & & \downarrow{\scriptstyle \alpha_1 \uplus \alpha_2} \\
\mathbb{E}_k(\mathcal{B}) & \xrightarrow{\mathbb{E}_k(h)} & \mathbb{E}_k(\mathcal{A}_1 \uplus \mathcal{A}_2) & \xrightarrow{\kappa} & \mathbb{E}_k(\mathcal{A}_1) \uplus \mathbb{E}_k(\mathcal{A}_2)
\end{array}$$

So we can rephrase our axioms about the lifted operation $\hat{\uplus}$ into conditions about the unlifted operation $\uplus$:

(S1') If $\uplus$ preserves embeddings, then $\hat{\uplus}$ preserves embeddings.

(S2') For every $(P, \pi) \in \mathcal{P}$, $(\mathcal{A}_i, \alpha_i) \in \mathbf{EM}(\mathbb{E}_k)$ and $f \colon P \to \mathcal{A}_1 \uplus \mathcal{A}_2$ such the following diagram commutes:

$$\begin{array}{ccccc} P & & \xrightarrow{\quad f \quad} & & \mathcal{A}_1 \uplus \mathcal{A}_2 \\ \downarrow \pi & & & & \downarrow \alpha_1 \uplus \alpha_2 \\ \mathbb{E}_k(P) & \xrightarrow{\mathbb{E}_k(f)} & \mathbb{E}_k(\mathcal{A}_1 \uplus \mathcal{A}_2) & \xrightarrow{\kappa} & \mathbb{E}_k(\mathcal{A}_1) \uplus \mathbb{E}_k(\mathcal{A}_2) \end{array} \tag{1}$$

then f has minimal decomposition as $f = e_1 \uplus e_2 \circ e_0$ where $e_i \colon (P_i, \pi_i) \rightarrowtail (\mathcal{A}_i, \alpha_i)$

Theorem

Given n-ary operation H that preserves embeddings, comonads $\mathbb{C}_1, \ldots, \mathbb{C}_n, \mathbb{D}$ *capturing* $\mathcal{J}_1, \ldots, \mathcal{J}_n, \mathcal{J}$ *and* $\kappa\colon \mathbb{D}(H(\mathcal{A}_1, \ldots, \mathcal{A}_n)) \to H(\mathbb{C}_1(\mathcal{A}_1), \ldots, \mathbb{C}_n(\mathcal{A}_n))$ *satisfying a similar diagram:*

$$\mathcal{A}_i \equiv_{\mathcal{J}_i} \mathcal{B}_i \textit{ implies } H(\mathcal{A}_1, \ldots, \mathcal{A}_n) \equiv_{\mathcal{J}} H(\mathcal{B}_1, \ldots, B_n)$$

To add equality, we consider a functor $\mathfrak{t}^I \colon \mathcal{R}(\sigma) \to \mathcal{R}(\sigma^I)$ where σ^I has additional binary relation I and $\mathfrak{t}^I(\mathcal{A})$ interprets $I^{\mathfrak{t}^I(\mathcal{A})}$ as equality on $\mathcal{A} \in \mathcal{R}(\sigma)$

Consider $\mathbb{E}_k \circ \mathfrak{t}^I \colon \mathcal{R}(\sigma) \to \mathcal{R}(\sigma^I)$ as a relative comonad over $\mathfrak{t}^I$.

As $\mathfrak{t}^I(\mathcal{A}_1 \uplus \mathcal{A}_2) \cong \mathfrak{t}^I(\mathcal{A}_1) \uplus \mathfrak{t}^I(\mathcal{A}_2)$

Study other enrichments such as first-order logic with a connectivity relation **conn** by considering a $\mathfrak{t}^{\mathbf{conn}}$

Products are easier since right adjoints, such as the cofree-coalgebra functor, preserve limits!

Many other comonads to explore:

- k-variable logic (Abramsky+Dawar+Wang 17)
- modal logic graded by depth
- guarded logics (Abramsky+Marsden 20)
- hybrid/bounded logics (Abramsky+Marsden 21)
- logics with generalised quantifiers (O'Conghaile+Dawar 20)
- logics with restricted conjunction (Montacute+S 22)

All of these are examples of arboreal covers which are studied axiomatically in Abramsky+Reggio 21